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Irregular Sampling of Bandlimited L^p-functions

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It is shown that bandlimited L^p-functions can be reconstructed by sampling series of Lagrange type with knots $\{t_n\}_{n \in \mathbb{Z}}$ whenever $|t_n - n| \leqslant L < \min\{1/2p, \frac{1}{4}\}$.

1. INTRODUCTION

Let B_β^p denote the space of all $L^p(\mathbf{R})$-functions that are bandlimited to $[-\beta, \beta]$ (see Section 2). The classical Whittaker–Shannon–Kotel'nikov sampling theorem [16, 17, 12, 5] states that

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} \qquad (z \in \mathbf{C}), \tag{1.1}$$

i.e., the function f can be reconstructed from its values at the integers, provided that $f \in B_\beta^p$ for $1 \leqslant p < \infty$, $\beta \leqslant \pi$ or $p = \infty$, $\beta < \pi$. Setting $t_n := n$, $G(z) := \pi^{-1} \sin \pi z$, formula (1.1) can we rewritten as

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z-t_n)} \qquad (z \in \mathbf{C}). \tag{1.2}$$

The function G can be interpreted as a canonical product with respect to the integers (cf. Section 3), i.e.,

$$G(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z}{t_n}\right)\left(1 - \frac{z}{t_{-n}}\right). \tag{1.3}$$

Hence in view of (1.2), (1.3) it is justified to call the sampling series (1.1) a Lagrange interpolation formula with infinitely many knots (cf. [10]).

The classical sampling theorem is often referred to as the uniform sampling theorem because the underlying sequence of knots (the sequence

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of integers) is equidistantly spaced. Nonuniform, or irregular, sampling theory investigates which not necessarily equidistantly spaced sequences of sampling knots admit the reconstruction of bandlimited functions; in particular, it is asked under which assumptions on f and $\{t_n\}_{n\in\mathbf{Z}}$ do the formulas (1.2), (1.3) remain valid.

The most common condition used for describing nonuniform sampling sequences is

$$|t_n - n| \leqslant L \qquad (n \in \mathbf{Z}) \text{ for some } L \geqslant 0. \tag{1.4}$$

In two papers, Higgins developed criteria that guarantee the validity of (1.2), (1.3). In [6] he uses Hilbert space techniques to prove that (1.2), (1.3) hold, if $p=2$, $L<\frac{1}{4}$, and in [7] he could show by function theoretic means that one may choose $L<1/4p$, provided $1<p\leqslant 2$.

In the present paper Higgins' results are extended to all spaces B_π^p, $1\leqslant p<\infty$, and a better bound for L is established as well. Indeed, (1.2) holds whenever.

$$L < \begin{cases} \frac{1}{4}, & 1\leqslant p\leqslant 2 \\ 1/2p, & 2\leqslant p<\infty. \end{cases}$$

This assertion is established in Section 4. The proof is based on an inequality due to Korevaar and a sharpened version of some estimates of Levinson (see Section 3). Section 5 deals with the absolute convergence of the irregular sampling series; an inequality that compares the l^p-norm of $\{f(t_n)\}_{n\in\mathbf{Z}}$ with the L^p-norm of f, given by Nikol'skiĭ for uniform sequences $\{t_n\}_{n\in\mathbf{Z}}$, is generalized to the nonuniform case. In Section 6 the new results are applied to two modifications of the usual sampling process, namely derivative sampling and oversampling.

Condition (1.4) is not the only condition that can be used to describe irregular sampling sequences. Two other types of nonuniform sequences also lead to interesting sampling series; first, sequences of type $t_n = n + D$, $t_{-n} = -t_n$, $n\in\mathbf{N}$, $t_0 = 0$ and, secondly, sequences that can be constructed by merging several (equidistant) sequences ("periodic sampling"). These types are investigated in [9, 3], respectively. A unified approach to all these types of nonuniform sampling can be found in [8] (my doctoral thesis). The most general result of [8] is cited at the end of Section 6.

2. Preliminaries

Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ denote the sets of natural, integer, real, and complex numbers, respectively. For $x\in\mathbf{R}$, the floor function $\lfloor x\rfloor$ is defined to be the largest integer $\leqslant x$.

Let $E \subset \mathbf{C}$ and let f_1, f_2 be real-valued, nonnegative functions on E. Then f_1 is *equivalent* to f_2 on E ($f_1 \sim f_2$ on E) if and only if there exist C_1, C_2 such that $0 < C_1 \leqslant C_2$ and $C_1 f_1(z) \leqslant f_2(z) \leqslant C_2 f_1(z)$ for $z \in E$. Sometimes, the well known, o, $\mathcal{O}$-notation will be used as well.

For $1 \leqslant p < \infty$, $L^p = L^p(\mathbf{R})$ denotes the space of all Lebesgue measurable functions f on $\mathbf{R}$ with $\|f\|_p := (\int_{-\infty}^{\infty} |f(x)|^p \, dx)^{1/p} < \infty$, and l^p is the space of all sequences $\{t\} = \{t_n\}_{n \in \mathbf{Z}}$ of finite norm $\|\{t\}\|_p := (\sum_{n \in \mathbf{Z}} |t_n|^p)^{1/p}$. As usual, $L^\infty(\mathbf{R})$ consists of all functions which are essentially bounded on $\mathbf{R}$.

The spaces B_β^p, $1 \leqslant p \leqslant \infty$, $\beta \geqslant 0$ are made up of all entire functions f which are in L^p when restricted to $\mathbf{R}$ and fullfill $|f(x+iy)| \leqslant \sup_{u \in \mathbf{R}} |f(u)| \, e^{\beta|y|}$, $x + iy \in \mathbf{C}$. If a function belongs to the space B_β^p, it is called bandlimited to $[-\beta, \beta]$, since its (distributional) Fourier transform vanishes outside of $[-\beta, \beta]$. Bandlimited functions can be estimated with the help of several remarkable inequalities.

LEMMA 2.1. *Let* $1 \leqslant p < \infty$, $\beta \geqslant 0$, *and* $f \in B_\beta^p$.

(a) (*Korevaar's inequality*)

$$|f(z)| \leqslant C \cdot \|f\|_p (1 + |y|)^{-1/p} e^{\beta|y|} \quad (z = x + iy \in \mathbf{C}),$$

(b) (*Bernstein's inequality*) $\|f'\|_p \leqslant \beta \cdot \|f\|_p$.

Proof. (a) See [11]. (b) Compare [14, p. 115]. ∎

In the calculations of this paper some formulas concerning the Gamma function are needed, namely the *functional equation* $\Gamma(z+1) = z\Gamma(z)$ ($z \in \mathbf{C} \setminus \{-n; n \in \mathbf{N} \cup \{0\}\}$), the *reflection formula* $1/(\Gamma(z)\,\Gamma(1-z)) = \pi^{-1} \sin \pi z$, $z \in \mathbf{C}$, and the estimates given in the following lemma.

LEMMA 2.2. (a) *Let* $\alpha, \beta \in \mathbf{R}$ *and* $\eta > 0$. *Then there holds*

$$\left| \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \right| \sim |z|^{\alpha - \beta}$$

on

$$\{z = x + iy \in \mathbf{C}; |z| \geqslant \eta, (x + \alpha) \geqslant \eta, (x + \beta) \geqslant \eta\}.$$

(b) *Let* $a_1, a_2, b_1, b_2 \in \mathbf{C}$ *with* $a_1 + a_2 - b_1 - b_2 = 0$. *Then*

$$\lim_{M \to \infty} \frac{\Gamma(M + a_1)\,\Gamma(M + a_2)}{\Gamma(M + b_1)\,\Gamma(M + b_2)} = 1.$$

Proof. A partial proof can be found in [13], a detailed one is given in [8, pp. 11–14]. ∎

3. Estimates of canonical products

DEFINITION 3.1. (a) A sequence $\{t\} = \{t_n\}_{n \in \mathbf{Z}}$ of real numbers is called a *perturbed equidistant sequence* (with respect to the sequence of integers), if there exists a constant $L \geqslant 0$ such that

$$|t_n - n| \leqslant L \qquad (n \in \mathbf{Z}).$$

(b) The *canonical product* with respect to a perturbed equidistant sequence $\{t\}$ is defined by

$$G(z) := G(\{t\}; z) := g(t_0; z) \prod_{k=1}^{\infty} g(t_k; z)\, g(t_{-k}; z) \qquad (z \in \mathbf{C}),$$

where $g(s; z) := 1 - z/s$, if $s \in \mathbf{R} \setminus \{0\}$ and $g(0; z) := z$.

Since $|g(t_k; z)\, g(t_{-k}; z) - 1| = \mathcal{O}(k^{-2})$ for large k, the product $G(z)$ is well defined and represents an entire function with zeros exactly at the points t_n, $n \in \mathbf{Z}$. The canonical product of Definition 3.1(b) is closely related to the canonical product investigated in function theory [1, p. 18; 8, p. 25]. In sampling theory, $G(z)$ is often defined with $z - t_0$ instead of $g(t_0; z)$.

PROPOSITION 3.1. *Let $\{t\}$ be a perturbed equidistant sequence with $L < \frac{1}{2}$ and $t_0 = 0$. Then there are constants C_1, C_2 such that for all $z = x + iy \in \mathbf{C}$ with $|z|$ large,*

$$C_1 H_1(z)\, H_2(L; z) \leqslant |G(z)| \leqslant C_2 H_1(z)\, H_2(-L; z), \tag{3.1}$$

where

$$H_1(z) := e^{\pi|y|} \begin{cases} 1, & |\mathcal{I}(z)| > 1 \\ \prod_{k=N}^{N+2} |t_k - z|, & |\mathcal{I}(z)| \leqslant 1 \text{ and } \mathcal{R}(z) > 0, \\ \prod_{-N-2}^{-N} |t_k - z|, & |\mathcal{I}(z)| \leqslant 1 \text{ and } \mathcal{R}(z) < 0 \end{cases}$$

$$H_2(d; z) := \begin{cases} |z|^{-4d}, & 0 \leqslant |\sin\theta| \leqslant \sin(\pi/2|z|) \\ |z|^{-2d} |\sin\theta|^{2d}, & \sin(\pi/2|z|) < |\sin\theta| \leqslant 1 \end{cases}$$

($N = N(z)$ is a suitable index to be defined below, and $d = L$, $-L$, respectively).

Proof. To begin with, assume that $\theta = \operatorname{Arg} z \in [0, \pi/2)$ and, say, $|z| \geqslant 4$. Defining, for those z,

$$N := N(z) := \max\{n \in \mathbf{N};\, n + L \leqslant |z|/\cos\theta\} = \lfloor |z|/\cos\theta - L \rfloor, \tag{3.2}$$

one has, in view of $t_0 = 0$ and $L < \frac{1}{2}$,

$$|G(z)| = |z| \prod_{k=1}^{N-1} \left|1 - \frac{z}{t_k}\right| \left|1 - \frac{z}{t_{-k}}\right| \prod_{k=N}^{N+2} \frac{|t_k - z|}{|t_k|} \left|1 - \frac{z}{t_{-k}}\right| \\ \times \prod_{k=N+3}^{\infty} \left|1 - \frac{z}{t_k}\right| \left|1 - \frac{z}{t_{-k}}\right|.$$

A straightforward calculation shows that $|g(s; z)| = |1 - z/s|$, $s \in \mathbf{R}\setminus\{0\}$, as function of s, decreases on $(0, |z|/\cos\theta)$ and increases everywhere else on $\mathbf{R}\setminus\{0\}$ (see [13, p. 56, Figure 2] for a geometric interpretation). Applying this fact and Definition 3.1(a), one obtains

$$\left|1 + \frac{z}{k+L}\right| \leqslant \left|1 - \frac{z}{t_{-k}}\right| \leqslant \left|1 + \frac{z}{k-L}\right| \qquad (k \in \mathbf{N}),$$

$$\left|1 - \frac{z}{k+L}\right| \leqslant \left|1 - \frac{z}{t_k}\right| \leqslant \left|1 - \frac{z}{k-L}\right| \qquad (1 \leqslant k < N),$$

$$\frac{1}{k+L} \leqslant \frac{1}{|t_k|} \leqslant \frac{1}{k-L} \qquad (N \leqslant k \leqslant N+2),$$

$$\left|1 - \frac{z}{k-L}\right| \leqslant \left|1 - \frac{z}{t_k}\right| \leqslant \left|1 - \frac{z}{k+L}\right| \qquad (k > N+2).$$

Hence one deduces,

$$\prod_{k=N}^{N+2} |t_k - z|\, H(L; z) \leqslant |G(z)| \leqslant \prod_{k=N}^{N+2} |t_k - z|\, H(-L; z) \\ (\theta \in [0, \pi/2),\ |z| \geqslant 4),$$

$$H(d; z) := |z| \prod_{k=1}^{N-1} \left|1 - \frac{z}{k+d}\right| \left|1 + \frac{z}{k+d}\right| \prod_{k=N}^{N+2} \frac{1}{k+d} \left|1 + \frac{z}{k+d}\right| \\ \times \prod_{k=N+3}^{\infty} \left|1 - \frac{z}{k-d}\right| \left|1 + \frac{z}{k+d}\right|.$$

By the functional equation of the gamma function, Lemma 2.2(b), and the reflection formula, one has, provided $d\in[-L, L]$ and $z\neq k+d$ for all $k\in\mathbf{N}$,

$$\begin{aligned}
H(d; z) &= \Bigg| z \prod_{k=1}^{N-1} (k+d-z) \prod_{k=1}^{N+2} \frac{k+d+z}{(k+d)^2} \\
&\quad \times \lim_{M\to\infty} \prod_{k=N+3}^{\infty} \frac{(k-d-z)(k+d+z)}{(k-d)(k+d)} \Bigg| \\
&= \Bigg| z \prod_{k=1}^{N-1} (k+d-z) \\
&\quad \times \frac{\Gamma(N+3+d+z)(\Gamma(1+d))^2\, \Gamma(N+3-d)}{\Gamma(1+d+z)(\Gamma(N+3+d))^2\, \Gamma(N+3-d-z)} \\
&\quad \times \frac{\Gamma(N+3+d)}{\Gamma(N+3+d+z)} \\
&\quad \lim_{M\to\infty} \frac{\Gamma(M+1-d-z)\, \Gamma(M+1+d+z)}{\Gamma(M+1-d)\, \Gamma(M+1+d)} \Bigg| \\
&= |z| \, \Bigg| \frac{\Gamma(N+d-z)\, \Gamma(z-d)}{\Gamma(1-(z-d))\, \Gamma(z-d)} \\
&\quad \times \frac{\Gamma(N+3-d)\, \Gamma^2(1+d)}{\Gamma(N+3+d)\, \Gamma(1+d+z)\, \Gamma(N+3-d-z)} \Bigg| \\
&= |z| \, \Gamma^2(1+d) \left| \frac{\Gamma(z-d)}{\Gamma(z+1+d)} \right| \\
&\quad \times \left| \frac{\Gamma(N+3-d)}{\Gamma(N+3+d)} \right| \left| \frac{\Gamma(N+3+d-z)}{\Gamma(N+3-d-z)} \right| \\
&\quad \times \left| \frac{\sin \pi(z-d)}{\pi \prod_{k=0}^{2} (N+k+d-z)} \right|. \qquad (3.3)
\end{aligned}$$

Formula (3.3) contains three quotients of gamma functions which can be estimated by help of Lemma 2.2(a); in fact,

$$\left| \frac{\Gamma(z-d)}{\Gamma(z+1+d)} \right| = \frac{1}{|z-d|} \left| \frac{\Gamma(z+1-d)}{\Gamma(z+1+d)} \right| \sim |z|^{-1-2d},$$

$$\left| \frac{\Gamma(N+3-d)}{\Gamma(N+3+d)} \right| \sim N^{-2d},$$

$$\left| \frac{\Gamma(N+3+d-z)}{\Gamma(N+3-d-z)} \right| \sim (N+3-L-z|^{2d}.$$

In view of (3.2), $N \sim |z|/\cos\theta$ and $N+L \leqslant |z|/\cos\theta < N+L+1$, which implies that $|z| = (N+L)\cos\theta + \xi_1 \cos\theta$ for some $\xi_1 \in [0, 1)$. Thus, setting $\xi_2 := 3 - \xi_1 - 2L$,

$$
\begin{aligned}
|N+3-L-z| &= \left|\frac{|z|}{\cos\theta} - L - \xi_1 + 3 - L - |z|(\cos\theta + i\sin\theta)\right| \\
&= \sqrt{\left(\frac{|z|}{\cos\theta} + \xi_2 - |z|\cos\theta\right)^2 + (|z|\sin\theta)^2} \\
&= \sqrt{|z|^2 \tan^2\theta \left(1 + 2\xi_2 \frac{\cos\theta}{|z|}\right) + \xi_2^2}.
\end{aligned}
$$

Note that $\xi_2 \in (1, 3]$, $(1 + 2\xi_2 \cos\theta/|z|) \in (1, 5/2)$. If $0 \leqslant \theta \leqslant \pi/(2|z|)$, then $\tan\theta = \sin\theta/\cos\theta \leqslant (\pi/(2|z|))/\cos(\pi/4) \leqslant \pi\xi_2/(\sqrt{2}|z|)$, and hence one has in this case $|z|^2 \tan^2\theta(1 + 2\xi_2\cos\theta/|z|) \leqslant \frac{5}{4}\pi^2\xi_2^2$. But is $\pi/(2|z|) < \theta < \pi/2$, then $\tan\theta \geqslant \sin\theta \geqslant 2\theta/\pi \geqslant 2\xi_2/(5|z|)$, and $\xi_2^2 \leqslant \frac{25}{9}|z|^2 \tan^2\theta(1 + 2\xi_2\cos\theta/|z|)$. One obtains

$$
|N+3-L-z| \sim \begin{cases} \xi_2 \sim 1, & 0 \leqslant \theta \leqslant \pi/(2|z|) \\ |z|\tan\theta, & \pi/(2|z|) < \theta < \pi/2 \end{cases} \tag{3.4}
$$

and, as a consequence,

$$
\begin{aligned}
&|z|\, \Gamma^2(1+d) \left|\frac{\Gamma(z-d)}{\Gamma(z+1+d)}\right| \left|\frac{\Gamma(N+3-d)}{\Gamma(N+3+d)}\right| \left|\frac{\Gamma(N+3+d-z)}{\Gamma(N+3-d-z)}\right| \\
&\quad\sim |z|^{-4d} (\cos\theta)^{2d} \begin{cases} 1, & 0 \leqslant \theta \leqslant \pi/(2|z|) \\ |z|^{2d}(\tan\theta)^{2d}, & \pi/(2|z|) < \theta < \pi/2 \end{cases} \\
&\quad\sim \begin{cases} |z|^{-4d}, & 0 \leqslant \theta \leqslant \pi/(2|z|) \\ |z|^{-2d}(\sin\theta)^{2d}, & \pi/(2|z|) < \theta < \pi/2 \end{cases} \sim H_2(d; z).
\end{aligned}
$$

It has still to be shown that

$$
\prod_{k=N}^{N+2} |t_k - z| \left|\frac{\sin\pi(z-d)}{\pi \prod_{k=0}^{2} (N+k+d-z)}\right| \sim H_1(z).
$$

Let $\mathscr{I}(z) > 1$. Then $|\sin(z-d)| \sim \exp(\pi y)$, since $|\sin\pi(z-d)|^2 = \sin^2\pi(x-d) + \sinh^2\pi y$ and $\sin^2\pi(x-d) \in [0, 1]$ as well as $\sinh^2\pi y = (1 - e^{-2\pi y})^2 \exp(2\pi y)/4 \sim e^{2\pi y}$. In addition,

$$
\frac{\prod_{k=N}^{N+2} |t_k - z|}{\prod_{k=0}^{2} (N+k+d-z)} = \prod_{k=0}^{2} \left|\frac{(N-z) + (t_{N+k} - N)}{(N-z) + (k+d)}\right| \sim 1
$$

since $|t_{N+k}-N|\leqslant 2+L$, $|k+d|\leqslant 2+L$ and $|(N-z)+(t_{N+k}-N)|\geqslant |\mathscr{I}(z)|>1$, $|(N-z)+k+d|>1$. Let $0\leqslant \mathscr{I}(z)\leqslant 1$. Then

$$\left|\frac{\sin \pi(z-d)}{\pi \prod_{k=0}^{2}(N+k+d-z)}\right| = \left|\frac{\sin \pi(z-N-d)}{\pi \prod_{k=0}^{2}(z-N-d-k)}\right| \sim 1$$

since, on the one hand, $\sin \pi\omega/(\pi \prod_{k=0}^{2}(\omega-k))$ is a nonvanishing, continuous function on $[-\frac{1}{3}, 2]\times[0,1]$ and, on the other hand, $\mathscr{R}(z-N-d)\in[-1/3, 2]$, $\mathscr{I}(z-N-d)\in[0,1]$, since $\mathscr{R}(z)\geqslant\sqrt{15}>3$, provided $\mathscr{I}(z)\in[0,1]$, $\mathscr{R}(z)>0$ and $|z|>4$, and after plugging in (3.2),

$$N+L-\tfrac{1}{3}\leqslant \mathscr{R}(z)=\frac{|z|}{\cos\theta}-\frac{\mathscr{I}^2(z)}{\mathscr{R}(z)}\leqslant N+L+1.$$

So far, inequality (3.1) is proved for all $z\in\mathbf{C}$ with $|z|\geqslant 4$, $\theta\in[0,\pi/2)$ and $z\neq k\pm L$ for all $k\in\mathbf{N}$. The case $z=k\pm L$ can be settled by explicit calculation of $\prod_{k=1}^{N-1}$ in terms of factorials. The estimates on the upper part of the imaginary axis are very similar to those presented above except that there is no need to introduce an index N. To complete the proof, note the symmetry properties of G,

$$G(\{t\};\bar{z})=\overline{G(\{t\};z)},$$
$$G(\{t\};-z)=-G(\{t^*\};z),$$

where $t_n^*=-t_{-n}$, $n\in\mathbf{Z}$; evidently $|t_n^*-n|\leqslant L<\frac{1}{2}$ and $t_0^*=0$. ∎

Remark 3.1. A much more detailed proof, which is also valid for certain other types of nonuniform sequences, is given in [8, pp. 29–51]. Parts of the above proof are due to Levinson [13, pp. 56–57]; the essential improvement of the results obtained by Levinson is the equivalence relation (3.4).

4. The Nonuniform Sampling Theorem

Definition 4.1. Let $\{t\}$ be a perturbed equidistant sequence with $L<1/2$, $t_0=0$, and G the corresponding canonical product. The nth reconstruction function Ψ_n, $n\in\mathbf{Z}$, is given by

$$\Psi_n(z):=\begin{cases}\dfrac{G(z)}{G'(t_n)(z-t_n)}, & z\in\mathbf{C}\setminus\{t_n\}\\[2ex] 1, & z=t_n.\end{cases}$$

If $\{t\}$ is the sequence of integers, then $\Psi_n(z) = \operatorname{sinc}(z-n)$. Note that the choice of $\Psi_n(t_n)$ makes Ψ_n an entire function with the "interpolatory property" $\Psi_n(t_m) = \delta_{nm}$ for $m, n \in \mathbf{Z}$.

THEOREM 4.1. *Let $\{t\}$ be a perturbed equidistant sequence, $t_0 = 0$ and G the canonical product with respect to $\{t\}$. If $1 \leqslant p < \infty$ and*

$$L < \begin{cases} \frac{1}{4}, & 1 \leqslant p \leqslant 2 \\ 1/2p, & 2 \leqslant p < \infty, \end{cases} \tag{4.1}$$

then for all $f \in B_\pi^p$,

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n)\, \Psi_n(z) \tag{4.2}$$

uniformly on each bounded subset B of the complex plane.

Proof. Consider the positively oriented Jordan curves $S_{l,m}$ defined by

$$\begin{aligned} S_{l,m} &:= \{R_m e^{i\theta}; -\pi/2 < \theta < \pi/2\} \cup [R_m i, -R_{-l} i] \\ &\quad \cup \{-R_{-l} e^{i\theta}; \pi/2 < \theta < 3\pi/2\} \cup [R_{-l} i, -R_m i] \; (m \in \mathbf{N} \cup \{0\}, l \in \mathbf{N}). \end{aligned}$$

where $R_n = n + \frac{1}{2}$, $n \in \mathbf{Z}$, i.e., the contour $S_{l,m}$ consists of two semicircles with radii R_m, $-R_{-l}$ and those parts of the imaginary axis which connect the endpoints of the semicircles. The numbers R_n are chosen in such a way that $t_n < R_n < t_{n+1}$, $n \in \mathbf{Z}$, and $|R_n - t_m| \geqslant \frac{1}{2} - L$ for all $m, n \in \mathbf{Z}$, $m \neq n$.

An application of the residue theorem yields for all $l \in \mathbf{N}$, $m \in \mathbf{N} \cup \{0\}$ and $z \in \operatorname{int}(S_{l,m})$ with $z \neq t_n$, $n \in \mathbf{Z}$,

$$\begin{aligned} \varepsilon_{l,m}(z) &:= \frac{G(z)}{2\pi i} \int_{S_{l,m}} \frac{f(\zeta)}{G(\zeta)(\zeta - z)}\, d\zeta \\ &= G(z) \left(\operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; z \right) \right. \\ &\quad \left. + \sum_{\substack{n \text{ such that} \\ R_{-l} < t_n < R_m}} \operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; t_n \right) \right) \\ &= f(z) - \sum_{n=-l+1}^{m} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)} \end{aligned} \tag{4.3}$$

The interpolatory property of the functions Ψ_n implies that $f(z) = \sum_{n=-l+1}^{m} f(t_n)\, \Psi_n(z)$ simultaneously for all $z = t_n$ in B, provided m, l are large enough. Thus, if $\varepsilon_{l,m}(z)$ vanishes uniformly on B (which will be shown in the rest of the proof), the assertion of the theorem is a consequence of (4.3).

By application of Lemma 2.1(a) and Proposition 3.1, it is possible to estimate $f(z)$ from above and $G(z)$ from below on the countours $S_{l,m}$. Indeed, for any $R>0$, $\theta\in(-\pi/2,\pi/2)\cup(\pi/2,3\pi/2)$ and for any $y\in\mathbf{R}$, respectively, one has

$$|f(Re^{i\theta})|\leqslant C\|f\|_p(1+|R\sin\theta|)^{-1/p}\,e^{\pi|R\sin\theta|}, \tag{4.4}$$

$$|f(yi)|\leqslant C\|f\|_p(1+|y|)^{-1/p}\,e^{\pi|y|}. \tag{4.5}$$

If $\theta\in(-\pi/2,\pi/2)\cup(\pi/2,3\pi/2)$ and R is large enough

$$|G(Re^{i\theta})|\geqslant CH_1(Re^{i\theta})\,H_2(L;Re^{i\theta}), \tag{4.6}$$

$$H_2(L;Re^{i\theta})=\begin{cases}R^{-4L}, & 0\leqslant|\sin\theta|\leqslant\sin(\pi/2R)\\ R^{-2L}|\sin\theta|^{2L}, & \sin\pi/2R<|\sin\theta|\leqslant 1.\end{cases} \tag{4.7}$$

The growth behaviour of H_1 on the semicircles of the contours $S_{l,m}$ is given by

$$H_1(R_me^{i\theta})\geqslant Ce^{\pi|R_n\sin\theta|}\qquad(m\in\mathbf{N}\text{ large},\ -\pi/2<\theta<\pi/2), \tag{4.8}$$

$$H_1(-R_{-l}e^{i\theta})\leqslant Ce^{\pi|R_{-l}\sin\theta|}\qquad(l\in\mathbf{N}\text{ large},\ \pi/2<\theta<3\pi/2). \tag{4.9}$$

These inequalities are a consequence of the definition of H_1, given in Proposition 3.1; in fact, in case of $-\pi/2<\theta<\pi/2$ (the other case is analogous) one has, noting that $\mathscr{R}(z)>0$, $R_m>0$, $t_N\geqslant 0$,

$$\prod_{k=N}^{N+2}|t_k-R_me^{i\theta}|\geqslant\prod_{k=N}^{N+2}|t_k-R_m|\geqslant(1/2-L)^3.$$

On the imaginary axis a corresponding estimate can be established provided $|y|$ is large enough, namely,

$$|G(yi)|\geqslant Ce^{\pi|y|}\,|y|^{-2L}. \tag{4.10}$$

Now one can estimate the contour integral. Obvious substitutions lead to the representation

$$\begin{aligned}\varepsilon_{l,m}(z)=\frac{G(z)}{2\pi i}\Bigg\{&\int_{-\pi/2}^{\pi/2}\frac{f(R_me^{i\theta})\,iR_me^{i\theta}}{G(R_me^{i\theta})(R_me^{i\theta}-z)}\,d\theta\\ &+\int_{R_m}^{-R_{-l}}\frac{f(yi)i}{G(yi)(yi-z)}\,dy+\int_{\pi/2}^{3\pi/2}\frac{f(-R_{-l}e^{i\theta})(-iR_{-l})\,e^{i\theta}}{G(-R_{-l}e^{i\theta})(-R_{-l}e^{i\theta}-z)}\,d\theta\\ &+\int_{R_{-l}}^{-R_m}\frac{f(yi)i}{G(yi)(yi-z)}\,dy\Bigg\}.\end{aligned}$$

Assuming that $z \in B$ and l, m are large, $|R_m e^{i\theta} - z| \geqslant ||R_m| - |z|| \geqslant R_m - \sup_{z \in B}|z| \geqslant CR_m$, and likewise $|-R_{-l}e^{i\theta} - z| \geqslant C|R_{-l}|$. Thus one obtains, applying (4.4)–(4.10),

$$|\varepsilon_{l,m}(z)| \leqslant C\|f\|_p \Bigg[R_m^{4L} \int_{\pi/(2R_m)}^{\pi/(2R_m)} (1+|R_m \sin\theta|)^{-1/p}\,d\theta$$

$$+ R_m^{2L} \left\{ \int_{\pi/2}^{\pi/(2R_m)} + \int_{\pi/(2R_m)}^{\pi/2} \right\} \frac{(1+|R_m \sin\theta|)^{-1/p}}{|\sin\theta|^{2L}}\,d\theta$$

$$+ |R_{-l}|^{4L} \int_{\pi+\pi/(2R_{-l})}^{\pi-\pi/(2R_{-l})} (1+|R_{-l}\sin\theta|)^{-1/p}\,d\theta$$

$$+ |R_{-l}|^{2L} \left\{ \int_{\pi/2}^{\pi+\pi/(2R_{-l})} + \int_{\pi-\pi/(2R_{-l})}^{3\pi/2} \right\} \frac{(1+|R_{-l}\sin\theta|)^{-1/p}}{|\sin\theta|^{2L}}\,d\theta$$

$$+ \left\{ \int_{\min\{R_m, -R_{-l}\}}^{\max\{R_m, -R_{-l}\}} + \int_{\min\{R_{-l}, -R_m\}}^{\max\{R_{-l}, -R_m\}} \right\} \frac{(1+|y|)^{-1/p}}{|y|^{1-2L}}\,dy \Bigg]$$

$$\leqslant C\|f\|_p\{R_m^{4L-1} + |R_{-l}|^{4L-1}\} \tag{4.11}$$

$$+ C\|f\|_p \begin{cases} R_m^{4L-1} + |R_{-l}|^{4L-1}, \\ \qquad 2L + 1/p > 1 \\ R_m^{4L-1}\log R_m + |R_{-l}|^{4L-1}\log|R_{-l}|, \\ \qquad 2L + 1/p = 1 \\ R_m^{2L-1/p} + |R_{-l}|^{2L-1/p}, \\ \qquad 2L + 1/p < 1 \end{cases} \tag{4.12}$$

$$+ C\|f\|_p \int_{\min\{R_m, -R_{-l}\}}^{\infty} |y|^{-1+(2L-1/p)}\,dy. \tag{4.13}$$

The expressions (4.11)–(4.13) become arbitrarily small as $m, l \to \infty$, since $L < \frac{1}{4}$ and $L < 1/(2p)$ (i.e., $2L - 1/p < 0$); in particular, all the terms in (4.12) vanish if and only if both conditions are fulfilled. Thus, $\varepsilon_{l,m}(z)$ tends to zero uniformly if $l, m \to 0$. This completes the proof. ∎

Remark 4.1. Theorem 4.1 contains a uniqueness criterion for band-limited functions, namely, let $\{t\}$ and p be given as in Theorem 4.1, and assume that $f_1, f_2 \in B_\pi^p$ such that $f_1(t_n) = f_2(t_n)$, $n \in \mathbf{Z}$. Then $f_1 \equiv f_2$ (on $\mathbf{C}$). For other uniqueness criteria for entire functions see, e.g., [2].

Remark 4.2. The assumption (4.1) may be weakened to the effect that $\{t\}$ is a strictly increasing sequence with $t_0 = 0$ and

$$|t_n - n| \leqslant \min\left\{\frac{1}{2p}, \frac{1}{4}\right\} \qquad (|n| \geqslant n_0)$$

for a suitable $n_0 \in \mathbf{N}$ (see [8] for details). An important example are sequences which approach the integers (i.e., $|t_n - n| = o(1)$, $|n| \to \infty$), arising quite naturally, e.g., from Sturm–Liouville boundary value problems (cf. [18]). Here it is sometimes even possible to express the reconstruction functions in terms of well known standard functions (cf. [3, 6, 18]).

The proof of Theorem 4.1 gives rise to an estimate of the so-called truncation error.

COROLLARY 4.1. *Let $(T_n f)(z)$ denote the truncation error, i.e.,*

$$(T_n f)(z) := f(z) - \sum_{n=-N}^{N} f(t_n)\, \Psi_n(z).$$

Under the same assumptions as made in Theorem 4.1, *B denoting an arbitrary bounded subset of* $\mathbf{C}$, *thereholds*

$$\sup_{z \in B} |(T_n f)(z)| \leqslant C \cdot \|f\|_p \begin{cases} N^{4L-1}, & p < 1/(1-2L) \\ N^{4L-1} \log N, & p = 1/(1-2L) \\ N^{2L-1/p}, & p > 1/(1-2L). \end{cases}$$

Proof. The estimate follows from (4.3) together with (4.11)–(4.13), noting that $R_n = n + \frac{1}{2}$ and that $1 - 4L > 1/p - 2L$ if and only if $p > 1/(1-2L)$. ∎

5. ABSOLUTE CONVERGENCE

In some applications as, e.g., the generalization of the sampling theorem to more than one dimension (cf. [4]), it is important that the cardinal series is absolutely and hence unconditionally convergent. This cannot be concluded from Theorem 4.1; another method of proof has to be chosen. To start with, an inequality, given by Nikol'skiĭ in the case of the uniform knots [14, pp. 123–124], will be generalized to a class of nonuniform sampling sequences which contains all the perturbed equidistant sequences with $L < \frac{1}{2}$.

THEOREM 5.1. *Let $1 \leqslant p < \infty$, and let $\{t\}$ be a sequence such that*

$$0 < \delta \leqslant \delta_n := t_{n+1} - t_n \leqslant L < \infty \qquad (n \in \mathbf{Z})$$

for suitable constants δ, L. Then, for all $f \in B_\pi^p$,

$$L^{-1/p}\|f\|_p \leqslant \sup_{x \in \mathbf{R}} \left(\sum_{n=-\infty}^{\infty} |f(t_n - x)|^p \right)^{1/p} \leqslant \delta^{-1/p}(1 + L\pi)\|f\|_p. \tag{5.1}$$

Proof. Applying the monotone convergence theorem, one obtains

$$\begin{aligned}\int_{-\infty}^{\infty} & |f(u)|^p \, du \\ &= \sum_{n=-\infty}^{\infty} \int_0^{\delta_{n-1}} |f(t_n - x)|^p \, dx \leqslant \sum_{n=-\infty}^{\infty} \int_0^L |f(t_n - x)|^p \, dx \\ &= \int_0^L \sum_{n=-\infty}^{\infty} |f(t_n - x)|^p \, dx \leqslant L \cdot \sup_{x \in \mathbf{R}} \sum_{n=-\infty}^{\infty} |f(t_n - x)|^p.\end{aligned}$$

This establishes the left inequality in (5.1). In view of the mean value theorem and the monotone convergence theorem,

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx = \sum_{n=-\infty}^{\infty} \int_{t_n}^{t_{n+1}} |f(x)|^p \, dx = \sum_{n=-\infty}^{\infty} \delta_n |f(\xi_n)|^p \tag{5.2}$$

for a certain sequence $\{\xi\}$ with $t_n < \xi_n < t_{n+1}$, $n \in \mathbf{Z}$. By Hölder's inequality, one has for $1 < p < \infty$ (q denoting the conjugate index)

$$\begin{aligned}\delta_n & \left| \int_{t_n}^{\xi_n} f'(x) \, dx \right|^p \\ &\leqslant \delta_n \left(\int_{t_n}^{t_{n+1}} |f'(x)| \, dx \right)^p \\ &\leqslant \delta_n \left(\int_{t_n}^{t_{n+1}} |f'(x)|^p \, dx \right) \delta_n^{p/q} = \delta_n^p \int_{t_n}^{t_{n+1}} |f'(x)|^p \, dx.\end{aligned}$$

Of course, this inequality is also true for $p = 1$. Using it, together with the fundamental theorem of the calculus, the monotone convergence theorem and Bernstein's inequality, one can conclude

$$\begin{aligned}&\left(\sum_{n=-\infty}^{\infty} \delta_n |f(\xi_n) - f(t_n)|^p \right)^{1/p} \\ &\quad = \left(\sum_{n=-\infty}^{\infty} \delta_n \left| \int_{t_n}^{\xi_n} f'(x) \, dx \right|^p \right)^{1/p} \leqslant \left(\sum_{n=-\infty}^{\infty} \delta_n^p \int_{t_n}^{t_{n+1}} |f'(x)|^p \, dx \right)^{1/p} \\ &\quad \leqslant \left(\sum_{n=-\infty}^{\infty} L^p \int_{t_n}^{t_{n+1}} |f'(x)|^p \, dx \right)^{1/p} \\ &\quad = L \|f'\|_p \leqslant L\pi \|f\|_p.\end{aligned} \tag{5.3}$$

The formulas (5.2) and (5.3) mean that $\{\delta_n^{1/p} f(\xi_n)\}_{n\in\mathbf{Z}}$, $\{\delta_n^{1/p} f(\xi_n) - \delta_n^{1/p} f(t_n)\}_{n\in\mathbf{Z}} \in l^p$; hence $\{\delta_n^{1/p} f(t_n)\}_{n\in\mathbf{Z}} \in l^p$ and

$$\left(\sum_{n=-\infty}^{\infty} |f(t_n)|^p\right)^{1/p} \leqslant \left(\sum_{n=-\infty}^{\infty} \frac{\delta_n}{\delta} |f(t_n|^p\right)^{1/p}$$
$$\leqslant \delta^{-1/p} \left\{\left(\sum_{n=-\infty}^{\infty} \delta_n |f(\xi_n) - f(t_n)|^p\right)^{1/p} + \|f\|_p\right\}$$
$$\leqslant \delta^{-1/p}(1 + L\sigma)\|f\|_p.$$

The right inequality in (5.1) follows, noting that the above argument remains valid if the sequence $\{t\}$ is replaced by any of the sequences $\{t_n - x\}_{n\in\mathbf{Z}}$, $x \in \mathbf{R}$. ∎

Remark 5.1. Any perturbed equidistant sequence $\{t\}$ with $|t_n - n| \leqslant L < \frac{1}{2}$, $n \in \mathbf{Z}$ satisfies the assumption of Theorem 5.1 since $0 < 1 - 2L \leqslant t_{n+1} - t_n \leqslant 1 + 2L$.

THEOREM 5.2. *Let $1 \leqslant p < \infty$, $\{t\}$ be a perturbed equidistant sequence with $t_0 = 0$, and*

$$L \leqslant \tfrac{1}{4}, \qquad p = 1$$
$$L < 1/4p, \qquad 1 < p < \infty.$$

Then for all $f \in B_\pi^p$ the series

$$\sum_{n=-\infty}^{\infty} \left| f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)} \right|$$

converges uniformly on each bounded subset of $\mathbf{C}$.

Proof. If B denotes an arbitrary bounded subset of $\mathbf{C}$, $C_B := \sup_{z\in B}|z|$, and if $n_0 \in \mathbf{N}$ is suitably chosen, one has

$$|z - t_n| \geqslant C \cdot |n| \qquad (|n| \geqslant n_0,\ z \in B), \tag{5.4}$$
$$|G'(t_n)| \geqslant C \cdot |n|^{-4L} \qquad (|n| \geqslant n_0), \tag{5.5}$$
$$|G(z)| \leqslant C \qquad (z \in B). \tag{5.6}$$

Inequality (5.4) holds with $n_0 \geqslant 2C_B + \frac{1}{2}$, noting that $|z - t_n| \geqslant |t_n| - C_B \geqslant |t_n|/2$. As to (5.5), in view of the symmetry properties of G it is no loss of generality to assume that n is positive. Noting that $N(z)=n$ in the interval $I_n := (n-L, n+L)$ and that t_n is a cluster point of I_n, one obtains, taking into account the estimate of $G(z)$ from below given in Proposition 3.1

$$|G'(t_n)| = \lim_{\substack{x \to t_n \\ \text{within } I_n}} \left|\frac{G(x)}{x-t_n}\right| \geqslant \liminf_{\substack{x \to t_n \\ \text{within } I_n}} x^{-4L}(t_{n+1}-x)(t_{n+2}-x)$$
$$\geqslant (1-2L)(2-2L)|t_n|^{-4L} \geqslant C \cdot |n|^{-4L}.$$

Inequality (5.6) is an immediate consequence of the fact that $G(z)$ is entire, hence continuous.

Let $m > l \geqslant n_0$. By Hölder's inequality (with q as conjugate index of p), (5.4)–(5.6) and Theorem 5.1,

$$R_{l,m}(z) := \sum_{l \leqslant |n| \leqslant m} \left| f(t_n) \frac{G(z)}{G'(t_n)(z-t_n)} \right|$$

$$\leqslant \begin{cases} \displaystyle\sum_{l \leqslant |n| \leqslant m} |f(t_n)| \sup_{l \leqslant |n| \leqslant m} \left|\frac{G(z)}{G'(t_n)(z-t_n)}\right|, & p=1 \\ \displaystyle\left(\sum_{l \leqslant |n| \leqslant m} |f(t_n)|^p\right)^{1/p} \left(\sum_{l \leqslant |n| \leqslant m} \left|\frac{G(z)}{G'(t_n)(z-t_n)}\right|^q\right)^{1/q}, & 1 < p < \infty \end{cases}$$

$$\leqslant \begin{cases} \displaystyle C \sum_{l \leqslant |n| \leqslant m} |f(t_n)| \sup_{|n| \geqslant n_0} \{|n|^{4L-1}\}, & p=1 \\ \displaystyle C\|f\|_p \left(\sum_{l \leqslant |n| \leqslant m} |n|^{(4L-1)q}\right)^{1/q}, & 1 < p < \infty. \end{cases}$$

In case of $p=1$, $\sum_{l \leqslant |n| \leqslant m} |f(t_n)|$ vanishes as $l \to \infty$, since $\{f(t_n)\}_{n \in \mathbf{Z}} \in l^1$ by Nikol'skiĭ's inequality. The second factor $\sup_{|n| \geqslant n_0} |n|^{4L-1}$ is bounded whenever $L \leqslant \frac{1}{4}$. If $1 < p < \infty$ and $L < 1/4p$, then $(4L-1)q < (1/p-1)q = -q/q = -1$ and $\sum_{l \leqslant |n| \leqslant m} |n|^{(4L-1)q}$ becomes arbitrarily small with l tending to infinity. Thus $R_{l,m}(z)$ vanishes uniformly on B as $l \to \infty$. By Cauchy's convergence criterion, this proves the assertion of the theorem. ∎

6. Modifications and Generalizations

This section contains two variants and a far-reaching generalization of Theorem 4.1. The variants deal with derivative sampling and with over-sampling.

THEOREM 6.1 (Derivative Sampling). *Let $r \in \mathbf{N}$, $\{t\}$ be a perturbed equidistant sequence with $t_0 = 0$, and G the canonical product with respect to $\{t\}$. If $1 \leqslant p < \infty$ and*

$$L < \begin{cases} \dfrac{1}{4(r+1)}, & 1 \leqslant p \leqslant 2 \\ \dfrac{1}{2p(r+1)}, & 2 \leqslant p < \infty, \end{cases}$$

there holds for all $f \in B^p_{\pi(r+1)}$, uniformly on each bounded subset of $\mathbf{C}$,

$$f(z) = \sum_{n=-\infty}^{\infty} \sum_{i=0}^{r} f^{(i)}(t_n)\, \psi_{r,n,i}(z)$$
$$\psi_{r,n,i}(z) := \sum_{\mu=0}^{r-i} \frac{G^{r+1}(z)}{(z-t_n)^{\mu+1}} \frac{\{((\cdot - t_n)/G(\cdot))^{r+1}\}^{(r-i-\mu)}(t_n)}{i!\,(r-i-\mu)!}. \tag{6.1}$$

Proof. With $S_{l,m}$ as in the proof of Theorem 4.1, define

$$\varepsilon^{(r)}_{l,m}(z) := \frac{G^{r+1}(z)}{2\pi i} \int_{S_{l,m}} \frac{f(\zeta)}{G^{r+1}(\zeta)(\zeta - z)}\, d\zeta.$$

Plugging in Korevaar's inequality to estimate $f \in B^p_{\pi(r+1)}$ from above and Proposition 3.1 for a lower bound G^{r+1} one obtains, on the one hand, arguing as in the proof of Theorem 4.1, that $\varepsilon^{(r)}_{l,m}(z) \to 0$, $l, m \to \infty$ uniformly on each bounded subset of $\mathbf{C}$. On the other hand, one calculates for $z \in \operatorname{int}(S_{l,m})$ and $z \neq t_n$, $n \in \mathbf{Z}$,

$$\varepsilon^{(r)}_{l,m}(z) = f(z) - \sum_{n=-l+1}^{m} G^{r+1}(z) \operatorname{Res}\left(\frac{f(\cdot)}{G^{r+1}(\cdot)(z-\cdot)}; t_n\right)$$
$$= f(z) - \sum_{n=-l+1}^{m} \frac{G^{r+1}(z)}{r!} \left(f(\zeta)(z-\zeta)^{-1}\left(\frac{\zeta - t_n}{G(\zeta)}\right)^{r+1}\right)^{(r)}\Bigg|_{\zeta = t_n}$$

and (6.1) follows by double application of Leibniz' rule, noting that $d^\mu/d\zeta^\mu((z-\zeta)^{-1}) = \mu!(z-\zeta)^{-1-\mu}$. ∎

EXAMPLE 6.1. If $r = 1$, the function f is reconstructed from the values of f and f' at the knots t_n, $n \in \mathbf{Z}$, and

$$\psi_{1,n,1}(z) = \frac{G^2(z)}{(G'(t_n))^2(z-t_n)},$$
$$\psi_{1,n,0}(z) = \Psi_{1,n,1}(z)\left\{\frac{1}{z-t_n} - \frac{G''(t_n)}{G'(t_n)}\right\}.$$

Remark 6.1. Higgins has established a similar result [7]; however, he needs to assume that $1 < p \leqslant 2$ and $L < 1/4p(r+1)$.

THEOREM 6.2. (Oversampling). *Let $\{t\}$ be a perturbed equidistant sequence with $t_0 = 0$, and let G be the canonical product with respect to $\{t\}$. If $L < \frac{1}{4}$, $1 \leqslant p \leqslant \infty$, and $f \in B_\beta^p$ for some $\beta < \pi$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)} \tag{6.2}$$

uniformly on each bounded subset of $\mathbf{C}$.

Proof. This sampling theorem can be proved in the same way as Theorem 4.1. The proof is omitted here since Seip [15] and Higgins [7] obtained the same result in this case (although their estimate of G was less precise). ∎

Remark 6.2. The assumption "$L < \frac{1}{4}$" in Theorem 6.2 is due to the convergence properties of the series (6.2). One can show (cf. [9, 8]) that the series (6.2) does not converge for arbitrary L. Nevertheless, the functions $f \in B_\beta^p$, $\beta < \pi$ are always uniquely determined by the sequence $\{f(t_n)\}_{n \in \mathbf{Z}}$, provided $\{t\}$ is a perturbed equidistant sequence with respect to the integers (see [8]).

All sequences known so far that give rise to a (nonuniform) sampling series are perturbed equidistant sequences (with respect to some equidistant sequence). However, sometimes the characterization $|t_n - n| \leqslant L$ is not precise enough; e.g., it does not show whether $|t_n| < |n|$, or whether there are subsequences $\{t_{n_j}\}_{j \in \mathbf{Z}}$ that fulfill a condition of type $|t_{n_j} - n_j| \leqslant L_1 < L$. In [8], a different way of describing nonuniform sequences is chosen. The following theorem states the main results of this study; it generalizes the results presented here.

THEOREM 6.3. *For $\tau \in \mathbf{R}$, $\sigma > 0$, $D \in \mathbf{R}$, and $\Delta \geqslant 0$ let* $\operatorname{pes}(\{\tau + \sigma\{\mathbf{Z}\}\}; D, \Delta)$ *denote the space of all sequences with*

$$\tau + \sigma n - \sigma D \leqslant t_n \leqslant \tau + \sigma n - \sigma(D - \Delta), \qquad n \leqslant -J$$

$$\tau + \sigma n + \sigma(D - \Delta) \leqslant t_n \leqslant \tau + \sigma n + \sigma D, \qquad n \geqslant J$$

for some $J \in \mathbf{N}$. Let $K \in \mathbf{N}$ and assume that $\{t\}$ is a sequence that can be constructed by merging suitable sequences $\{t_1\}, ..., \{t_K\}$ with $\{t_j\} \in$ $\operatorname{pes}(\{\tau_j + \sigma_j\{\mathbf{Z}\}\}; D_j, \Delta_j)$ $(1 \leqslant j \leqslant K)$. *Let $W := \sum_{j=1}^{K} \sigma_j^{-1}$, let $G(z)$ be the canonical product with respect to $\{t\}$ and let $\{s\} = \{s_n\}_{n \in \mathbf{Z}}$ be the sequence of zeros of G.*

If $1 \leqslant p \leqslant \infty$ *and*

(i) $\sum_{j=1}^{K} (2D_j + \Delta_j) < 1$,

(ii) $\sum_{j=1}^{K} D_j < 1/2p$ $(:= 0,$ if $p = \infty)$,

then there is a strictly increasing sequence $\{R_m\}_{m \in \mathbf{Z}}$ *with* $\lim_{m \to \pm\infty} R_m = \pm\infty$, *such that for all* $f \in B^p_{\pi W}$,

$$f(z) = \lim_{\substack{l \to \infty \\ m \to \infty}} \sum_{\substack{n \text{ with} \\ R_{-l} < s_n < R_m}} G(z) \operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(z - \cdot)}; s_n\right)$$

uniformly on $B \setminus \{s_n; n \in \mathbf{Z}\}$, *where* B *is an arbitrary bounded subset of* $\mathbf{C}$.

Proof. [8, *pp.* 56–61]. See also [9, 3]. ∎

Remark 6.3. If $\{t\}$ is defined as in Theorem 6.3, there may be members of $\{t\}$ that occur several times. Then G has a zero of corresponding multiplicity. It is always possible to express the residues in terms of derivative values of f and G, but then the corresponding formulas are rather unhandy. Theorem 6.3 can be interpreted as a generalization of Hermite's interpolation formula to the situation of infinitely many knots.

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